*Journal of Statistical Physics, Vol. 38, Nos. 5/6, 1985*

# The Equations of Wilson's Renormalization Group in Dimension 4 and Analytic Renormalization

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*Received July 3, 1984*

Wilson's renormalization group equations are introduced and investigated in the framework of perturbation theory with respect to the deviation of the renormalization exponent from its bifurcation value. We consider the case when the dimension is equal to 4. An exact solution of these equations is constructed using analytic renormalization of the projection Hamiltonians.

**KEY WORDS:** Renormalization group; $\varepsilon$ expansion; analytic renormalization; bifurcation.

## 1. INTRODUCTION

In two previous papers[1,2] we solved Wilson's renormalization group equations for an effective Hamiltonian whose free part is defined by long-range potential $U(x) \sim -\text{const}/|x|^a$, $|x| \to \infty$. Also it was assumed that the dimension $d$ is not divisible by 4. As is well known, there exists a Gaussian branch of fixed points of renormalization group, defined by the family of the Hamiltonians of the form

$$H_0 = \int_{|k| < \Lambda} |k|^{a-d} \, |\sigma(k)|^2 \, d^d k$$

There is such a discrete series of values of $a$ that the corresponding spectrum of the differential of renormalization group on the Gaussian branch contains eigenvalue equal to 1. We expect that by analogy with finite-dimensional theory of bifurcations for such series of parameter $a$ a new branch of non-Gaussian solutions bifurcates from the branch of

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0022-4715/85/0300-0851$04.50/0 

Gaussian fixed points. The multiplicity of eigenvalue 1 is equal to one in the case when the dimension $d$ is not divisible by 4, and multiplicity of eigenvalue 1 is equal to two in the case $d=4$ (we consider the most interesting point of bifurcation $a_0=\frac{3}{2}d$).

General theorems of nonlinear analysis predict the bifurcation of only one non-Gaussian branch in the first case, that was considered in the framework of perturbation theory in the work.[2] In the second case from the point of view of general theory of dynamical systems an answer is not so definite. We shall show that in the case of dimension 4 also appears only one branch, which also has a constructive representation, basing on the operation of analytic renormalization. One can try to construct a non-Gaussian solution of this new branch as a power series in the deviation of the parameter $a$ from the bifurcation value $a_0$:

$$a_0=\tfrac{3}{2}d, \qquad \varepsilon=a-\tfrac{3}{2}d$$

## 2. WILSON'S EQUATIONS

First of all we will give all relevant definitions and properties concerning renormalization group. A Hamiltonian in the ball $\Omega=\{k: |k|<R\}$ is an expression of the form

$$H(\sigma)=\sum_{m=1}^{\infty}\int_{\Omega^m} h_m(k_1,\dots,k_m)\,\delta(k_1+\cdots+k_m)\prod_{i=1}^{m}\sigma(k_i)\,d^dk_i$$

If for some $n$, $h_i\equiv 0$, when $i>n$, the Hamiltonian $H(\sigma)$ is called the finite particle. Two Hamiltonians are equivalent, if the sequences of their coefficient functions coincide on the subspaces $k_1+\cdots+k_m=0$. A formal Hamiltonian is a formal series in $\varepsilon$, $H=H_0+\varepsilon H_1+\cdots$ whose coefficients are finite-particle Hamiltonians. The space of formal Hamiltonians with infinitely smooth coefficient functions $h_m$, $m=1,2,\dots$ will be denoted by $\mathcal{F}\mathcal{H}^\infty$.

Wilson's renormalization transformation is a composition of two transformations, namely, the scaling $R^a_\lambda$ and the restriction $\mathcal{S}_{\Omega,\lambda}$.

The action of the scaling operator $R^a_\lambda$ on the $m$-particle Hamiltonian

$$H=\int_{\Omega^m} h_m(k_1,\dots,k_m)\,\delta(k_1+\cdots+k_m)\prod_{i=1}^{m}\sigma(k_i)\,d^dk_i$$

is given by the formula

$$R^a_\lambda H=\lambda^{am/2-md+d}\int_{(\lambda\Omega)^m}\left(\frac{k_1}{\lambda},\dots,\frac{k_m}{\lambda}\right)\delta(k_1+\cdots+k_m)\prod_{i=1}^{m}\sigma(k_i)\,d^dk_i \tag{2.1}$$

To the whole space of formal Hamiltonians the scaling operator is extended by linearity.

The restriction operator restricts the random Gibbsian field in the ball $\lambda\Omega$, with Hamiltonian $R_\lambda^a H$ to the subvolume $\Omega$. If $H_0 + H'$ is a Hamiltonian of random field in the ball $\lambda\Omega$, then the Hamiltonian of its restriction to the ball $\Omega$ has the form

$$\mathscr{S}_{\Omega,\lambda}(H' + H_0) = :\exp[H'(\sigma)]:^c_{-\Delta(\chi_\lambda - \chi)} + H_0 \tag{2.2}$$

$: :^c_{-\Delta(\chi_\lambda-\chi)}$ is the connected Wick ordering with respect to the free Gaussian measure with correlation function

$$\delta(k_1 + k_2)[-\Delta(\chi_\lambda - \chi)(k)], \qquad \Delta(\chi_\lambda - \chi)(k) = |k|^{d-a}\left[\chi\left(\frac{k}{\lambda}\right) - \chi(k)\right]$$

where $\chi(k) \in C_0^\infty(\mathbb{R}^d)$, $\chi(k)$ is the smoothed characteristic function of the ball $\Omega$.

The Wilson renormalization transformation is a composition of the scaling and restriction operators

$$R_{\chi,\lambda}^{(a)} \equiv R_{\Omega,\lambda}^{(a)} = \mathscr{S}_{\Omega,\lambda} R_\lambda^{(a)} \tag{2.3}$$

Transformations $R_{\Omega,\lambda}^{(a)}$ form a multiplicative group in $\lambda$. Let us introduce a topology in the space of Hamiltonians $\mathscr{F}\mathscr{H}^\infty$: $h^{(n)} = (h_1^{(n)}, h_2^{(n)},...) \to 0$ if there exists $N > 0$ such that $h_m^{(n)} \equiv 0$ for $m \geqslant N$, $h_m^{(n)} \to 0$ in the $C^\infty$ topology. Then for any $\lambda > 0$ the operator $R_{\chi,\lambda}^{(a)}$ is a nonlinear continuous infinitely differentiable (along any directions) mapping from $\mathscr{F}\mathscr{H}^\infty$ to $\mathscr{F}\mathscr{H}^\infty$, and, moreover is an entire function of the parameter $a$.

Wilson's equations arise when one looks for nontrivial fixed points of the renormalization transformation near the bifurcation points.

We expect that, as usual in many problems of nonlinear analysis, for certain values of the parameter $a$, a new branch of solutions bifurcates from the branch of Gaussian fixed points of the renormalization group (RG). Typically, this new branch is unique; however, several branches may arise in degenerate cases. One can try to construct non-Gaussian solutions on this new branch as power series in the deviation of the parameter $a$ from the bifurcation value $a_0$. Owing to the invariance of the Hamiltonian under the action of the RG a hierarchy of equations on the coefficients of this series arises which we call Wilson's hierarchy of equations. Thus we have the problem of evaluation of the bifurcation values.

So, let us denote

$$\mathscr{F}_2^0 = \mathscr{F}^0 \otimes_{\mathrm{C}} \mathscr{F}, \qquad \mathscr{F}_2^0 = \left\{ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} \delta^n \varepsilon^m \right\}$$

the space of formal power series in two variables,

$$\tau: \mathscr{F}_2 \to \mathscr{F}_2^0, \qquad \tau: \sum_{m,n} a_{mn} \delta^n \varepsilon^m \to \sum_{m,n} a_{mn} \varepsilon^{m+n}$$

is an operator of restriction to the diagonal, and let $\mathscr{F}_2\mathscr{H}^\infty = \mathscr{F}_2 \otimes \mathscr{H}^\infty$ be the space of formal Hamiltonians depending on two variables. The renormalization operator is an entire function of the parameter $a$, and therefore we can write

$$R_{\chi,\lambda}^{(a)} = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \frac{d^n}{da^n} R_{\chi,\lambda}^{(a_0)}$$

where all the operators $(d^n/da^n)\, R_{\chi,\lambda}^{(a)}$ are continuous in the space of formal Hamiltonians $\mathscr{F}\mathscr{H}^\infty$ and the series converges for any $\delta$.

We introduce a new operator

$$R_{\chi,\lambda}^{(a_0,\delta)}: \mathscr{F}\mathscr{H}^\infty \to \mathscr{F}_2\mathscr{H}^\infty$$

$$R_{\chi,\lambda}^{(a_0,\delta)}: \sum_{m=1}^{\infty} \varepsilon^m H_m \to \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \frac{d^n}{da^n} R_{\chi,\lambda}^{(a_0)} \left( \sum_m \varepsilon^m H_m \right)$$

**Definition.** $a_0$ is a bifurcation value, if there exists a formal Hamiltonian $H \in \mathscr{F}\mathscr{H}^\infty$ such

$$\tau R_{\chi,\lambda}^{(a_0,\delta)} H = H \tag{2.4}$$

The equality (4) is understood as the equality of formal Hamiltonians. In that case Hamiltonian $H$ is called an effective Hamiltonian.

As the general theory predicts, bifurcation points may be only such points, whose spectrum of differential of corresponding nonlinear transformation contains eigenvalue 1. Analysis of the spectrum of differential of renormalization groups was performed in Refs. 3 and 4. If dimension $d$ in not divisible by 4, then the spectrum of differential of RG in the first bifurcation value $a_0 = \frac{3}{2}d$ contains simple eigenvalue 1 with eigenvector

$$H_1 = \varphi^4 = \int \delta(k_1 + \cdots + k_4)\, \sigma(k_1) \cdots \sigma(k_4)\, dk_1 \cdots dk_4, \qquad \mathscr{D}_{\chi,\lambda}^{(a_0)} H_1 = H_1$$

where $\mathscr{D}_{\chi,\lambda}^{(a_0)} = \langle R_\lambda^{(a_0)} \cdot \rangle_{\Delta(\chi\lambda - \chi)}$ is a differential of renormalization transformation in 0 with $a = a_0 = \frac{3}{2}d$. Coefficient functions must satisfy the conditions of smoothness, oddness in the spin variable, symmetry, and isotropy. In Ref. 1 is shown that in such a case there exists a unique nontrivial branch of fixed points. In the case $d = 4$ in the spectrum of $\mathscr{D}_{\chi,\lambda}^{(a_0)}$

there arise two eigenvectors with eigenvalue 1, which we denote $\varphi^4$ and $(\nabla\varphi)^2$:

$$\mathscr{D}_{\chi,\lambda}^{(a_0)}\varphi^4=\varphi^4, \qquad \mathscr{D}_{\chi,\lambda}^{(a_0)}(\nabla\varphi)^2=(\nabla\varphi)^2$$

where

$$(\nabla\varphi)^2=\int k_1^2\delta(k_1+k_2)\,\sigma(k_1)\,\sigma(k_2)\,dk_1\,dk_2$$

In such a situation general theorems of the theory of bifurcation did not work: may bifurcate 0, 1, 2, 3, branches of fixed point (see Refs. 6 and 7).

We shall look for a solution in the class of projection Hamiltonians.

## 3. ANALYTIC RENORMALIZATION AND PROJECTION HAMILTONIANS

Let $\mathscr{F}\mathscr{H}_b^\infty$ be the space of formal Hamiltonians, whose coefficient functions are bounded at infinity together with all their derivatives. Hamiltonian $H\in\mathscr{F}\mathscr{H}_b^\infty$ of the form

$$H=:\exp\mathscr{L}:^c_{-\Delta(1-\chi)}$$

where $\mathscr{L}\in\mathscr{F}\mathscr{H}_b^\infty$ is called a projection Hamiltonian. Really, it is a Hamiltonian of the random Gibbsian field, which is obtained by restriction of a random field in the whole momentum space with Hamiltonian $\mathscr{L}$ to the ball $\Omega$.

The action of renormalization group in the space of projection Hamiltonians is simplified:

$$R_{\chi,\lambda}^{(a)}:\exp\mathscr{L}:^c_{-\Delta(1-\chi)}=:\exp(R_\lambda^{(a)}\mathscr{L}):^c_{-\Delta(1-\chi)} \tag{3.1}$$

The projection Hamiltonian $H$ is well defined when $\operatorname{Re}a>2d$, and admits an analytic continuation to the whole complex plane as a meromorphic function of $a$.

We shall look for a solution of Wilson's equations in the form

$$H=:\exp[u\varphi^4+v(\nabla\varphi)^2]:^c_{-\Delta(1-\chi)} \tag{3.2}$$

where $u$, $v$ are independent parameters. The bifurcation value $a_0=\frac{3}{2}d$ is a pole of the analytic continuation of Hamiltonian $H$. The operation of analytic renormalization is the most adequate procedure for regularization of such a Hamiltonian. We shall formulate some definitions concerning analytic renormalization.

Let $G$ be an arbitrary graph arising in the expansion over Feynman graphs and $\mathscr{F}_G$ the Feynman amplitude corresponding to the graph $G$. As we have said, $\mathscr{F}_G$ is a meromorphic function of $\varepsilon$. The renormalized amplitude is given by the formula

$$\text{A.R. } \mathscr{F}_G = \sum_{A \in A(G)} \mathscr{F}_{G:A}$$

where $A(G) = \{\{H_1, ..., H_k\} \mid H_i$ are pairwise disjoint one-particle irreducible subgraphs of $G\}$,

$$\mathscr{F}_{G:A} = (2\pi)^{-h} \int dk_1 \cdots dk_h \prod_{l \in L(G/A)} \Delta(1-\chi)(q_l) \prod_{H \in A} O(H)$$

$G/A$ is a quotient graph, $L(H)(E(H))$ is a set of internal (external) lines of $H$, $h$ is a Betti number of graph $G/A$, and let $\{q_l \mid l \in L(G/A)\}$ and $\{k_i \mid i = 1, 2, ..., h(G/A)\}$ be line and loop variables for the quotient graph $G/A$, $\{p_e \mid e \in E(G/A)\}$, the set of external momenta of $G/A$, and $O(H)$ is a vertex part for $H$.

Here are some properties of the vertex part $O(H)$:

(i) If $|E(H)| = 4$, then

$$O(H) = \sum_{n=1}^{|V(H)|-1} a_n^{(H)} \left(\frac{1}{\varepsilon}\right)^n$$

is a polynomial in $\varepsilon^{-1}$ of degree $|V(H)| - 1$, where $|V(H)|$ is a number of vertices of the graph $H$.

(ii) If $|E(H)| = 2$, then

$$O(H) = \left[\sum_{n=1}^{|V(H)|-1} a_n^{(H)} \left(\frac{1}{\varepsilon}\right)^n\right] p_e^2$$

where $e \in E(H)$.

(iii) If $|V(H)| = 1$, then $O(H) = 1$. If $H$ is not one-particle irreducible or $|E(H)| > 4$, then $O(H) \equiv 0$. $O(H)$ depends only on subgraph $H$.

**Theorem 1.** To every connected graph $H$ a vertex part $O(H)$ can be associated in such a way that the renormalized amplitude A.R. $\mathscr{F}_G$ is an analytic function of $\varepsilon$ in some neighborhood of the origin.

The proof of this important theorem can be derived from the analysis in Refs. 8 and 9 (see also Ref. 2). More deep and precise facts about vertex parts can be obtained from the so-called scaling relations for $O(H)$.

**Theorem 2.** (Introduction of counterterms):

$$\text{A.R.}{:}\exp[u\varphi^4 + v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)} = {:}\exp[w_1\varphi^4 + w_2(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)} \quad (3.3)$$

where

$$w_1(u, v) = \sum_{n=1, m=0}^{\infty} u^n v^m O_{n,m}, \qquad w_2(u, v) = \sum_{n=1, m=0}^{\infty} u^n v^m O'_{n,m} + v \quad (3.4)$$

Here

$$O_{n,m} = \sum_{k=1}^{n-1} \varepsilon^{-k} \sum_{G \in \mathscr{S}_{m,n}} a_k^{(G)}, \qquad O'_{n,m} = \sum_{k=1}^{n-1} \varepsilon^{-k} \sum_{G \in \mathscr{S}'_{m,n}} a_k^{(G)} \quad (3.5)$$

where $\mathscr{S}_{m,n}$ ($\mathscr{S}'_{m,n}$) is the set of all one-particle irreducible graphs with 4 (2) external lines, generated by $n$ vertices with 4 external lines and $m$ vertices with 2 external lines. The equality (3.3) is understood in the sense of formal series in $u$.

The proof of Theorem 2 is analogous to the proof of Theorem 3.3 of Ref. 2.

## 4. SOLUTION OF WILSON'S EQUATIONS

Let us define an action of renormalization group on the renormalized projection Hamiltonian.

The following result holds:

**Lemma 1.**

$$\begin{aligned} R^a_{\chi,\lambda}\, &\text{A.R.}{:}\exp[u\varphi^4 + v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)} \\ &= {:}\exp[\lambda^{2\varepsilon} w_1\varphi^4 + \lambda^{\varepsilon} w_2(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)} \end{aligned}$$

The proof of this result is easily derived from Theorem 2 and equalities $R_\lambda^{(a)}\varphi^4 = \lambda^{2\varepsilon}\varphi^4$, $R_\lambda^{(a)}(\nabla\varphi)^2 = \lambda^{\varepsilon}(\nabla\varphi)^2$.

Let us introduce a new variable $\tau$: $\lambda = \exp(\tau/2)$. Transformations $R^{(a)}_{\chi,\exp(\tau/2)}$ form an additive group of transformations (by $\tau$). In the following it will be convenient to use an infinitesimal operator of this group. Denote this operator as

$$W = \lim_{\tau \to 0} \frac{R^{(a)}_{\chi,\exp(\tau/2)} - I}{\tau}$$

**Lemma 2.**

$$W\,\text{A.R.}{:}\exp[u\varphi^4+v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$
$$=\left(\rho_1\frac{d}{du}+\rho_2\frac{d}{dv}\right)\text{A.R.}{:}\exp[u\varphi^4+v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$

where

$$\rho_1(u,v)=\varepsilon(w_1w'_{2,v}-\tfrac{1}{2}w_2w'_{1,v})(w'_{1,u}w'_{2,v}-w'_{2,u}w'_{1,v})^{-1} \tag{4.1}$$

$$\rho_2(u,v)=\varepsilon(\tfrac{1}{2}w_2w'_{1,u}-w_1w'_{2,u})(w'_{1,u}w'_{2,v}-w'_{2,u}w'_{1,v})^{-1} \tag{4.2}$$

*Proof.* It is directly checked that

$$W\,\text{A.R.}{:}\exp[u\varphi^4+v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$
$$={:}\exp[w_1\varphi^4+w_2(\nabla\varphi)^2][\varepsilon w_1\varphi^4+\varepsilon\tfrac{1}{2}w_2(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$

We turn up equalities

$$\frac{d}{du}\text{A.R.}{:}\exp[u\varphi^4+v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$
$$={:}[w'_{1,u}\varphi^4+w'_{2,u}(\nabla\varphi)^2]\exp[w_1\varphi^4+w_2(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$

$$\frac{d}{dv}\text{A.R.}{:}\exp[4\varphi^4+v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$
$$={:}[w'_{1,v}\varphi^4+w'_{2,v}(\nabla\varphi)^2]\exp[w_1\varphi^4+w_2(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)}$$

and using them in the preceding formula, we get the assertion of the lemma. ▮

**Theorem 3.** The formal series $\rho_1$, $\rho_2$ have the form

$$\rho_1(u,v)=\sum_{n=2,m=0}^{\infty}a_{nm}u^nv^m+\varepsilon u,\qquad \rho_2(u,v)=\sum_{n=2,m=0}^{\infty}b_{n,m}u^nv^m+\varepsilon v$$

where coefficients $a_{nm}$ and $b_{nm}$ do not depend on $\varepsilon$, $a_{20}\neq 0$.

*Proof.* By direct calculations one can see that

$$w_1(u,v)=u+\sum_{n=2}^{\infty}u^n\sum_{m=0}^{\infty}v^mO_{nm} \tag{4.3}$$

$$w_2(u,v)=v+\sum_{n=2}^{\infty}u^n\sum_{m=0}^{\infty}v^mO'_{nm} \tag{4.4}$$

where $O_{nm}$ and $O'_{nm}$ are polynomials in $\varepsilon^{-1}$ without constant terms. All graphs, spanned by one 4-line graph and nonzero number of 2-line graphs have detachable subgraphs and therefore terms of the type $uv^n$, $n=1, 2,...$ are absent in the expansions (4.3), (4.4). Vertex parts, corresponding to graphs, spanned by 2-line graphs, equal to 0 and therefore terms of the type $v^n$ are absent in (4.4). Substituting (4.3), (4.4) in (4.1), (4.2) we receive the expansions for series $\rho_1$ and $\rho_2$. Proof of independence of $a_{nm}$ and $b_{nm}$ of $\varepsilon$ may be obtained by induction in $n$ and $m$ and is analogous to proof of Theorem 4.1 in Ref. 2. We omit this part of proof. ▌

If we solve the equation

$$\left(\rho_1 \frac{d}{du} + \rho_2 \frac{d}{dv}\right) \text{A.R.:}\exp[u\varphi^4 + v(\nabla\varphi)^2]{:}^{c}_{-\Delta(1-\chi)} = 0 \tag{4.5}$$

we also solve Wilson's equations. This equation has to be solved in the formal series $u(\varepsilon) = u_1\varepsilon + u_2\varepsilon^2 + \cdots$, $v(\varepsilon) = v_1\varepsilon + v_2\varepsilon^2 + \cdots$.

The left side of Eq. (3) may be transformed to

$$\rho_1(u, v){:}\varphi^4{:}^{c}_{-\Delta(1-\chi)} + \rho_2(u, v){:}(\nabla\varphi)^2{:}^{c}_{-\Delta(1-\chi)}$$
$$+ \sum_{n,m=1}^{\infty} \text{A.R.:}(\varphi^4)^m((\nabla\varphi)^2)^n{:}^{c}_{-\Delta(1-\chi)} \frac{u^{n-1}v^{m-1}}{(n-1)!(m-1)!}\left(\frac{v}{m}\rho_1 + \frac{u}{n}\rho_2\right)$$

This series must be equal to 0 in all orders in $\varepsilon$ and therefore $\rho_1(u, v) = 0$, $\rho_2(u, v) = 0$. In fact, it is true in the first order in $\varepsilon$ because Hamiltonians

$${:}\varphi^4{:}^{c}_{-\Delta(1-\chi)} = \int \delta(k_1 + \cdots + k_4){:}\sigma(k_1)\cdots\sigma(k_4){:}^{c}_{-\Delta(1-\chi)}\, dk_1 \cdots dk_4$$

$${:}(\nabla\varphi)^2{:}^{c}_{-\Delta(1-\chi)} = \int \delta(k_1 + k_2){:}\sigma(k_1)\,\sigma(k_2){:}^{c}_{-\Delta(1-\chi)}\, k_1^2\, dk_1\, dk_2$$

are linear independent and all terms in the sum have degree $\varepsilon$ more than 1. Let $\rho_1(u, v) = 0$, $\rho_2(u, v) = 0$ in $k$ orders on $\varepsilon$. Using linear independence of Hamiltonians ${:}\varphi^4{:}^{c}_{-\Delta(1-\chi)}$, ${:}(\nabla\varphi)^2{:}^{c}_{-\Delta(1-\chi)}$, and that all terms in the sum have degree in $\varepsilon$ more than $k+1$ we obtain that $\rho_1 = 0$, $\rho_2 = 0$ in the $k+1$ order in $\varepsilon$.

So,

$$\rho_1(u, v) = 0, \qquad \rho_2(u, v) = 0$$

This system of equations has two solutions in formal power series $u = v \equiv 0$ and $u_1(\varepsilon)$, $v_1(\varepsilon)$. The first solution $u = v \equiv 0$ corresponds to the

Gaussian fixed point. The second, nontrivial non-Gaussian solution exists, because $a_2 \neq 0$. Really, from the first equation one can obtain the expression

$$u = \sum_{n=1}^{\infty} \varepsilon^n d_n(v), \qquad d_n(v) = \sum_{m=0}^{\infty} d_{m,n} v^m \tag{4.6}$$

where $d_n(v)$ are recurrently defined by $a_n(v)$. Substituting (4.6) in the second equation, we come to the equation

$$\sum_{k=2}^{\infty} \varepsilon^k c_k(v) + \varepsilon v = 0, \qquad c_k(v) = \sum_{l=0}^{\infty} c_{k,l} v^l$$

This equation has the unique solution in formal power series in $\varepsilon$ and corresponds to the non-Gaussian branch of fixed points.

It is easy to show that this branch is thermodynamically unstable.

## ACKNOWLEDGMENTS

The author is indebted to Dr. P. M. Bleher and Prof. Ya. G. Sinai for useful remarks.

## REFERENCES

1. P. M. Bleher and M. D. Missarov, *Commun. Math. Phys.* **74**:235–254 (1980).
2. P. M. Bleher and M. D. Missarov, *Commun. Math. Phys.* **74**:255–272 (1980).
3. Ya. G. Sinai, *Teor. Veroyatn. Ee Primen.* **21**:63–80 (1976).
4. P. M. Bleher, in *Multicomponent Random Systems* (in Russian) R. L. Dobrushin and Ya. G. Sinai, eds. (Nauka, Moscow, 1978).
5. Ya. G. Sinai, *Theory of Phase Transitions* (in Russian) (Nauka, Moscow, 1980).
6. J. B. Keller and S. Antman, eds. *Bifurcation Theory and Nonlinear Eigenvalue Problems* (Benjamin, New York, 1969).
7. L. Nirenberg, *Topics in Nonlinear Functional Analysis* (Courant Institute of Mathematical Sciences, New York, 1974).
8. E. Speer, in *Renormalization Theory*, G. Velo and A. S. Wightman, eds. (Reidel, Dordrecht, 1976), pp. 25–93.
9. E. Speer, *Commun. Math. Phys.* **23**:23–36 (1971).